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# A direct perturbation theory for dark solitons based on a complete set of the squared Jost solutions

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**Abstract.** Because of the essential difficulty caused by the non-vanishing boundary condition, a systematic perturbation approach for dark solitons has not yet been found. Based on a rigorous proof of the completeness of the squared Jost solution with explicit expressions for one-soliton case, a direct perturbation approach for dark solitons is developed in this paper. Difficulties caused by the background are overcome. As an example of the approach, the problem of damping is treated as a perturbation.

## 1. Introduction

Under ideal conditions, the well known nonlinear Schrödinger (NLS) equation

$$\mathrm{i}p_t - \sigma p_{xx} + 2|p|^2 p = 0 \tag{1}$$

governs the evolution of temporal optical solitons (see, e.g., [1] and references therein) in fibres and spatial optical solitons (see, e.g., [2, 18] and references therein) in waveguides. In the case of $\sigma = -1$, i.e. the case of the abnormal group-velocity dispersion (GVD) for temporal solitons or self-focusing media for spatial solitons, equation (1) has bright soliton solutions under vanishing boundary conditions. In the case of $\sigma = 1$, i.e. the case of the normal GVD or self-defocusing media, equation (1) has dark soliton solutions under non-vanishing boundary condition (with background). Higher-order effects are usually treated as small perturbations.

For a nonlinear evolution equation under a perturbation of strength $\epsilon$ ($0 < \epsilon \ll 1$), the zeroth-order approximation to the perturbed equation, as is well known, cannot be an exact soliton solution, otherwise the first-order correction will not remain small for a long time. In order to eliminate such so-called secular behaviour, one should allow free parameters in the exact soliton solution to modulate on the slow time scale $\epsilon t$ [10]. Thus the zeroth-order approximation is an adiabatic solution, where free parameters in the exact soliton solution evolve adiabatically following up the perturbations.

In the case of vanishing boundary, there exist two systematic perturbation methods: the method based upon IST (see, e.g., [7–9]) and the direct method (see, e.g., [10–15]) based upon the theory of linear partial differential equations. Another method based upon conservation laws (see, e.g., [16]) to get evolution of soliton parameters is also widely used but it should not be regarded as a systematic method, because no correction higher than the adiabatic solution can be obtained within its framework.

In what follows, we will first review the main strategies of the perturbation theories for vanishing boundary case, then the essential difficulties in generalizing them to the non-vanishing boundary case.

In the framework of IST, for a completely integrable nonlinear evolution equation, poles of the transmission coefficient are independent of time. However, in the presence of perturbations, they will depend on time. In the perturbation method based upon IST [7–9], the idea of an adiabatic solution is realized by keeping the first Lax equation, which yields the inverse scattering equations, and giving up the second one, which determines evolution of the scattering data. Instead of the abandoned second Lax equation, an inhomogeneous equation, whose homogeneous version is the first Lax equation is derived to consider the demands of the perturbed equation. Then its solutions can be represented by linear combination of the solutions of the first Lax equation, the Jost solutions. Owing to the vanishing boundary condition, the asymptotic behaviour of the Jost solutions yields the time dependence of the scattering coefficients and the spectral parameters.

Despite various formats used by different authors, the main strategy of the direct method [10–15] is

(i) To linearize the perturbed equation by multiple time scales. The first-order equation is a linear inhomogeneous equation of the first-order correction $q$, $Lq = R$.
(ii) To solve the eigenvalue problem of the linear operator $L$, $L\Phi = \lambda\Phi$, and the associated eigenvalue problem of its adjoint $L^{\mathrm{A}}$, $L^{\mathrm{A}}\Phi^{\mathrm{A}} = \lambda^{\mathrm{A}}\Phi^{\mathrm{A}}$.
(iii) With the adjoint eigenfunctions, the inner products can be defined. If a complete set of eigenfunctions can be constructed, $q$ can be expanded with the complete set.
(iv) Usually the discrete terms are secular. Thus the corresponding expansion coefficients must be set to zero in order to cancel these secular terms. These are the so-called secular conditions, governing the evolution of all free soliton parameters. The zeroth-order approximation, i.e. the adiabatic solution, is then obtained.
(v) After removing the secular terms, the sum of the remaining terms is the first-order correction $q$.

In general, as emphasized by some authors (see, e.g., [10, 15]), the method needs no knowledge of the IST except the general $N$-soliton solution. However, results of IST are very helpful in solving the eigenvalue problems. Usually $\Phi$ and $\Phi^{\mathrm{A}}$ can be constructed with the squared Jost solutions. Proof of the completeness of the set of squared Jost solutions, the foundation of the direct method, also depends on results of the IST. In the vanishing boundary case, completeness of the set of squared Jost solutions of Zakharov–Shabat eigen equation was proved under the assumption of compact support [6]. The direct perturbation theory (also called Green's function perturbation theory) has been well established for nearly integrable systems [10–15].

For the dark soliton case ($\sigma = 1$), let $p = u\mathrm{e}^{\mathrm{i}2\rho^2 t}$, in which $\rho$ is a positive constant, equation (1) becomes

$$\mathrm{i}u_t - u_{xx} + 2(|u|^2 - \rho^2)u = 0 \tag{2}$$

which is called the NLS$^+$ equation. Under non-vanishing boundary conditions

$$u \to \begin{cases} \rho & x \to +\infty \\ \rho\mathrm{e}^{\mathrm{i}\theta} & x \to -\infty \end{cases} \tag{3}$$

equation (2) has dark soliton solutions, e.g., the one-soliton solution:

$$u_1(x,t) = \mathrm{e}^{-\mathrm{i}\beta_1}\{\lambda_1 + \mathrm{i}k_1 \tanh\theta_1\} \tag{4}$$

in which

$$\theta_1 = k_1(x - x_1 - 2\lambda_1 t) \tag{5}$$

$$\zeta_1 = \lambda_1 + \mathrm{i}k_1 = \rho \mathrm{e}^{\mathrm{i}\beta_1}. \tag{6}$$

As mentioned above, in the presence of perturbation, poles of the transmission coefficient ($\zeta_1$ for one-soliton case) will be dependent of time. Then $\lambda_1$, $k_1$, $\rho$, $\beta_1$, and possibly $x_1$ are all $t$-dependent. The magnitude of the boundary condition values, $\rho$, and the phase difference between each end of the boundary values, $\theta$ ($\theta = -2\beta_1$ for one-soliton case), are therefore $t$-dependent as well. The evolution of the boundary condition values (the background) under perturbation is an essential difficulty. The existence of $\theta$ also makes it impossible for any attempt to separate the background from the NLS equation and directly generalize the perturbation theory for bright solitons to that for dark solitons.

Without vanishing boundary conditions, the usual deduction of the perturbation method based upon IST seems very difficult because the asymptotic behaviour of Jost solutions is related to the $t$-dependent non-vanishing boundary condition values (the background). There appears to have been no attempt to develop such a method up till now.

The direct method faces similar difficulties as the method based on IST:

(i) Without a vanishing boundary, it is hard to find the adjoint operator $L^{\mathrm{A}}$ and the corresponding eigenfunctions.

(ii) The potential in the Lax equations has no compact support. The proof of the completeness of the squared Jost solutions based on the assumption of compact support [6] is no longer valid.

(iii) Non-vanishing boundary means an infinite background energy which may cause divergence in calculations.

An earlier attempt at direct method [17] was proposed. However, since the evolution of the background, the most crucial and difficult point of the perturbation theory for dark solitons, is not dealt with at all in [17]; it is not successful in general. The obtained relations between small variations of the scattering data and the nonlinear field are obviously incorrect in view of the non-vanishing boundary condition. Hence the proof for completeness of the squared Jost solutions with these relations is also unfounded.

The adiabatic method [18, 19] based upon an attempt to separate the background from the NLS$^+$ equation has also been proposed. As mentioned above, such a separation is impossible because of the existence of $\theta$. Also, it has been found the method has a difficulty in self-consistency: results from the first perturbed conservation law contradicts that from the second and third ones [20]. Nevertheless, it is interesting that the formulae given by this method yields some correct final results, except that no shift of the soliton centre ($z_{\mathrm{c}}$ in this paper) can be predicted.

The small-amplitude approximation [5] transformed the NLS equation into a KdV equation in the small-amplitude limit, in order to use the mature perturbation theories for the KdV equation. However, considering the different symmetries between the NLS equation and KdV equation under inversion of $x$, the approximation is too drastic.

As a conclusion, a systematic perturbation theory has never been successfully developed for dark solitons. In this paper, we develop a direct perturbation theory for dark solitons. The three difficulties mentioned above are overcome by the following tricks:

(i) We find that it is appropriate to define the adjoint states and the corresponding inner products in a manner similar to that for bright solitons [6], because such a definition yields orthogonalities of the continuum squared Jost solutions and the completeness of the set of squared Jost solutions can be proved under such a definition.

(ii) We directly substitute the explicit expressions of the squared Jost solutions to prove completeness.
(iii) We find the divergent terms appearing in one of the secular conditions are all in the form of $2\mathcal{L}$ (the size of the system). This secular condition should yield two equations: one for the finite terms, the other for the $2\mathcal{L}$ terms. The latter governs the evolution of $\rho$, the magnitude of the background.

As a result, simple formulae for the evolution of all soliton parameters are presented. The formula for calculating the first-order correction is also obtained, though it is hard to get an exact analytic result. The problem of damping is studied as an example.

## 2. Some results of the NLS$^+$ equation

Dark soliton solution and Jost solutions of the NLS$^+$ equation (2) under non-vanishing boundary conditions (3) are well known [4, 21, 22], we just list the results which are helpful in constructing a direct perturbation theory. In what follows, the bar denotes complex conjugate and $\sigma_i$ $(i = 1, 2, 3)$ are Pauli matrices.

The corresponding Lax equations of (2) are

$$\partial_x \Phi(\lambda) = L(\lambda)\Phi(\lambda) \tag{7}$$

$$\partial_t \Phi(\lambda) = M(\lambda)\Phi(\lambda) \tag{8}$$

where the Lax pair is

$$L(\lambda) = -\mathrm{i}\lambda\sigma_3 + U \tag{9}$$

$$M(\lambda) = \mathrm{i}2\lambda^2\sigma_3 - 2\lambda U + \mathrm{i}\left(U^2 - \rho^2 + U_x\right)\sigma_3 \tag{10}$$

with

$$U = \begin{pmatrix} 0 & u \\ \bar{u} & 0 \end{pmatrix}. \tag{11}$$

Because a double valued function of $\lambda$

$$\kappa = \sqrt{\lambda^2 - \rho^2} \tag{12}$$

will appear in the asymptotic solutions of (7) as $|x| \to \infty$, an auxiliary parameter $\zeta$ can be introduced to make

$$\lambda = \tfrac{1}{2}\left(\zeta + \rho^2\zeta^{-1}\right) \qquad \text{and} \qquad \kappa = \tfrac{1}{2}\left(\zeta - \rho^2\zeta^{-1}\right) \tag{13}$$

become single-valued functions of $\zeta$ [21, 22]. With asymptotic solutions of (7), $E(x, \zeta)$ (as $x \to +\infty$) and $E_-(x, \zeta)$ (as $x \to -\infty$), usual Jost solutions are defined as

$$\Psi(x, \zeta) = \left(\tilde{\psi}(x, \zeta) \quad \psi(x, \zeta)\right) \to E(x, \zeta) \qquad \text{as } x \to \infty \tag{14}$$

and

$$\Phi(x, \zeta) = \left(\phi(x, \zeta) \quad \tilde{\phi}(x, \zeta)\right) \to E_-(x, \zeta) \qquad \text{as } x \to -\infty. \tag{15}$$

Here

$$E_-(x, \zeta) = \mathrm{e}^{\frac{1}{2}\mathrm{i}\theta\sigma_3} E(x, \zeta) \tag{16}$$

and

$$E(x, \zeta) = \begin{pmatrix} 1 & -\mathrm{i}\rho\zeta^{-1} \\ \mathrm{i}\rho\zeta^{-1} & 1 \end{pmatrix} \mathrm{e}^{-\mathrm{i}\kappa x\sigma_3}. \tag{17}$$

$\Phi(x, \zeta)$ and $\Psi(x, \zeta)$ are not linearly independent:

$$\phi(x, \zeta) = a(\zeta)\tilde{\psi}(x, \zeta) + b(\zeta)\psi(x, \zeta) \tag{18}$$

$$\tilde{\phi}(x, \zeta) = \tilde{b}(\zeta)\tilde{\psi}(x, \zeta) + \tilde{a}(\zeta)\psi(x, \zeta). \tag{19}$$

$\psi(x, \zeta)$, $\psi(x, \zeta)$ and $a(\zeta)$ can be analytically continued to the upper half-plane of complex $\zeta$, while $\tilde{\psi}(x, \zeta)$, $\tilde{\psi}(x, \zeta)$ and $\tilde{a}(\zeta)$ can be analytically continued to the lower half-plane of $\zeta$. Usually $b(\zeta)$ and $\tilde{b}(\zeta)$ cannot be analytically continued outside the real axes.

Since two values of $\zeta$ correspond to a single value of $\lambda$, under the transformation $\zeta \to \rho^2\zeta^{-1}$, the Jost solutions have the following so-called reduction relations:

$$\tilde{\psi}(x, \rho^2\zeta^{-1}) = \mathrm{i}\rho^{-1}\zeta\psi(x, \zeta) \qquad \psi(x, \rho^2\zeta^{-1}) = -\mathrm{i}\rho^{-1}\zeta\tilde{\psi}(x, \zeta) \tag{20}$$

$$\phi(x, \rho^2\zeta^{-1}) = \mathrm{i}\rho^{-1}\zeta\tilde{\phi}(x, \zeta) \qquad \tilde{\phi}(x, \rho^2\zeta^{-1}) = -\mathrm{i}\rho^{-1}\zeta\phi(x, \zeta). \tag{21}$$

Also

$$\tilde{a}(\rho^2\zeta^{-1}) = a(\zeta) \tag{22}$$

and

$$\tilde{b}(\rho^2\zeta^{-1}) = -b(\zeta) \qquad \zeta \text{ real.} \tag{23}$$

The zeros of $a(\zeta)$ are located on a circle of radius $\rho$ centred at the origin

$$\zeta_n = \rho\mathrm{e}^{\mathrm{i}\beta_n} \qquad 0 < \beta_n < \pi \tag{24}$$

or

$$\zeta_n = \lambda_n + \kappa_n \qquad \kappa_n = \mathrm{i}k_n \quad k_n > 0. \tag{25}$$

In the case of non-reflection

$$\phi(x, \zeta) = a(\zeta)\tilde{\psi}(x, \zeta) \qquad \tilde{\phi}(x, \zeta) = \tilde{a}(\zeta)\psi(x, \zeta) \tag{26}$$

$$\theta = -2\sum_{n=1}^{N}\beta_n \tag{27}$$

$$a(\zeta) = \mathrm{e}^{\frac{1}{2}\mathrm{i}\theta}\prod_{n=1}^{N}\frac{\zeta - \zeta_n}{\zeta - \bar{\zeta}_n}. \tag{28}$$

When $a(\zeta)$ has only one zero $\zeta_1$, the usual IST procedure yields the one-soliton solution (4) and the corresponding Jost solutions (see the appendix). In order to satisfy the second Lax equation(8), the Jost solutions should be corrected as

$$\phi(x, t, \zeta) \to h(t, \zeta)\phi(x, t, \zeta) \qquad \tilde{\phi}(x, t, \zeta) \to h^{-1}(t, \zeta)\tilde{\phi}(x, t, \zeta) \tag{29}$$

$$\tilde{\psi}(x, t, \zeta) \to h(t, \zeta)\tilde{\psi}(x, t, \zeta) \qquad \psi(x, t, \zeta) \to h^{-1}(t, \zeta)\psi(x, t, \zeta) \tag{30}$$

with

$$h(t, \zeta) = \mathrm{e}^{\mathrm{i}2\kappa\lambda t}. \tag{31}$$

## 3. The perturbed NLS$^+$ equation and the squared Jost solutions

The perturbed NLS$^+$ equation can be written as

$$\mathrm{i}v_t - v_{xx} + 2(|v|^2 - \rho^2)v = \epsilon r[v] \tag{32}$$

where $\epsilon$ is a small parameter and $r[v]$ is a functional of $v$ for most higher-order effects of dark solitons. Consider an approximate solution of (32) up to first order:

$$v = u + \epsilon q \tag{33}$$

with the initial condition

$$v(x, 0) = u(x, 0) \tag{34}$$

where $u(x, 0)$ is the exact soliton solution of the unperturbed NLS$^+$ at $t = 0$. According to the idea of adiabatic solutions, the zeroth-order approximation $u$ cannot be the exact solution of (2) or it may cause secular behaviour of the first-order correction $q$ [11]. In order to reduce those secular terms, parameters in $u$ must evolve with temporal scale of $(\epsilon t)$. Introducing the multiscale expansion

$$\partial_t = \sum_n^{\infty} \epsilon^n \partial_{t_n} \tag{35}$$

in which $t_n = \epsilon^n t$ $(n = 0, 1, 2, \ldots)$ are treated as independent variables as usual, to first order in $\epsilon$ we have

$$\mathrm{i}u_{t_0} - u_{xx} + 2(|u|^2 - \rho^2)u = 0 \tag{36}$$

and

$$\mathrm{i}q_{t_0} - q_{xx} + 2(2|u|^2 - \rho^2)q + 2u^2\overline{q} = R[u] \tag{37}$$

where $R[u] = r[u] - \mathrm{i}u'$ is the effective source (the prime means a derivative with respect to $t_1$). Having obtained an expression for the effective source, it is not necessary to distinguish $t_0$ and $t$. Together with the complex conjugate of (37), we have

$$\left\{\mathrm{i}\partial_t - \boldsymbol{L}(u)\right\}\boldsymbol{q} = \boldsymbol{R} \tag{38}$$

in which

$$\boldsymbol{L}(u) = \begin{pmatrix} \partial_{xx} - 2(2|u|^2 - \rho^2) & -2u^2 \\ 2\overline{u}^2 & -\partial_{xx} + 2(2|u|^2 - \rho^2) \end{pmatrix} \tag{39}$$

and

$$\boldsymbol{q} = \begin{pmatrix} q \\ \overline{q} \end{pmatrix} \qquad \boldsymbol{R} = \begin{pmatrix} R \\ -\overline{R} \end{pmatrix}. \tag{40}$$

Equation (38) is a linear inhomogeneous equation, whose solutions can be expressed by a linear combination of the solutions of its corresponding homogeneous equation

$$\{\mathrm{i}\partial_t - \boldsymbol{L}(u)\}\,\boldsymbol{q} = 0. \tag{41}$$

It is easy to show that if

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \tag{42}$$

is a solution of the unperturbed Lax equation (7) and (8), then

$$W(x, t, \zeta) = \begin{pmatrix} w_1^2(x, t, \zeta) \\ w_2^2(x, t, \zeta) \end{pmatrix} \tag{43}$$

is a solution of (41), and thus

$$h^2(t,\zeta)\begin{pmatrix}\phi_1^2(x,t,\zeta)\\ \phi_2^2(x,t,\zeta)\end{pmatrix} \qquad h^{-2}(t,\zeta)\begin{pmatrix}\tilde{\phi}_1^2(x,t,\zeta)\\ \tilde{\phi}_2^2(x,t,\zeta)\end{pmatrix} \tag{44}$$

$$h^2(t,\zeta)\begin{pmatrix}\tilde{\psi}_1^2(x,t,\zeta)\\ \tilde{\psi}_2^2(x,t,\zeta)\end{pmatrix} \qquad h^{-2}(t,\zeta)\begin{pmatrix}\psi_1^2(x,t,\zeta)\\ \psi_2^2(x,t,\zeta)\end{pmatrix} \tag{45}$$

are solutions of (41). Defining the squared Jost solutions as

$$\Phi(x,t,\zeta)=\begin{pmatrix}\phi_1^2(x,t,\zeta)\\ \phi_2^2(x,t,\zeta)\end{pmatrix} \qquad \tilde{\Phi}(x,t,\zeta)=\begin{pmatrix}\tilde{\phi}_1^2(x,t,\zeta)\\ \tilde{\phi}_2^2(x,t,\zeta)\end{pmatrix} \tag{46}$$

$$\tilde{\Psi}(x,t,\zeta)=\begin{pmatrix}\tilde{\psi}_1^2(x,t,\zeta)\\ \tilde{\psi}_2^2(x,t,\zeta)\end{pmatrix} \qquad \Psi(x,t,\zeta)=\begin{pmatrix}\psi_1^2(x,t,\zeta)\\ \psi_2^2(x,t,\zeta)\end{pmatrix} \tag{47}$$

we have

$$\left\{\mathrm{i}\partial_t - \boldsymbol{L}(u)\right\}\Phi(x,t,\zeta) = 4\kappa\lambda\Phi(x,t,\zeta) \tag{48}$$

$$\left\{\mathrm{i}\partial_t - \boldsymbol{L}(u)\right\}\tilde{\Phi}(x,t,\zeta) = -4\kappa\lambda\tilde{\Phi}(x,t,\zeta) \tag{49}$$

$$\left\{\mathrm{i}\partial_t - \boldsymbol{L}(u)\right\}\Psi(x,t,\zeta) = -4\kappa\lambda\Psi(x,t,\zeta) \tag{50}$$

$$\left\{\mathrm{i}\partial_t - \boldsymbol{L}(u)\right\}\tilde{\Psi}(x,t,\zeta) = 4\kappa\lambda\tilde{\Psi}(x,t,\zeta). \tag{51}$$

In the case of one soliton, it is convenient to discuss the problem in a tracing frame of reference (TFR) which is moving with the soliton, i.e. to make the transformation

$$t,\ x \quad\rightarrow\quad t,\ z \tag{52}$$

where

$$z = x - x_1 - 2\lambda_1 t. \tag{53}$$

Thus

$$\partial_t,\ \partial_x \quad\rightarrow\quad \partial_t - 2\lambda_1\partial_z,\ \partial_z. \tag{54}$$

In the TFR, we can define $t$-independent Jost solutions:

$$\phi(z,\zeta) = \mathrm{e}^{\mathrm{i}\kappa(x_1+2\lambda_1 t)}\phi(x,t,\zeta) \tag{55}$$

$$\tilde{\phi}(z,\zeta) = \mathrm{e}^{-\mathrm{i}\kappa(x_1+2\lambda_1 t)}\tilde{\phi}(x,t,\zeta) \tag{56}$$

$$\psi(z,\zeta) = \mathrm{e}^{-\mathrm{i}\kappa(x_1+2\lambda_1 t)}\psi(x,t,\zeta) \tag{57}$$

$$\tilde{\psi}(z,\zeta) = \mathrm{e}^{\mathrm{i}\kappa(x_1+2\lambda_1 t)}\tilde{\psi}(x,t,\zeta) \tag{58}$$

and the corresponding squared Jost solutions

$$\Phi(z,\zeta)=\begin{pmatrix}\phi_1^2(z,\zeta)\\ \phi_2^2(z,\zeta)\end{pmatrix} \qquad \tilde{\Phi}(z,\zeta)=\begin{pmatrix}\tilde{\phi}_1^2(z,\zeta)\\ \tilde{\phi}_2^2(z,\zeta)\end{pmatrix} \tag{59}$$

$$\Psi(z,\zeta)=\begin{pmatrix}\psi_1^2(z,\zeta)\\ \psi_2^2(z,\zeta)\end{pmatrix} \qquad \tilde{\Psi}(z,\zeta)=\begin{pmatrix}\tilde{\psi}_1^2(z,\zeta)\\ \tilde{\psi}_2^2(z,\zeta)\end{pmatrix}. \tag{60}$$

Then equations (48)–(51) become the following eigenequations:

$$\boldsymbol{L}(z)\Phi(z,\zeta) = -4\kappa(\lambda-\lambda_1)\Phi(z,\zeta) \tag{61}$$

$$\boldsymbol{L}(z)\tilde{\Phi}(z,\zeta) = 4\kappa(\lambda-\lambda_1)\tilde{\Phi}(z,\zeta) \tag{62}$$

$$\boldsymbol{L}(z)\Psi(z,\zeta) = 4\kappa(\lambda-\lambda_1)\Psi(z,\zeta) \tag{63}$$

$$\boldsymbol{L}(z)\tilde{\Psi}(z,\zeta) = -4\kappa(\lambda-\lambda_1)\tilde{\Psi}(z,\zeta). \tag{64}$$

These mean that the eigenfunctions of the linear operator $L$ have been constructed with the squared Jost solutions. At $\zeta_1$, we have

$$\boldsymbol{L}(z)\Phi(z,\zeta_1) = 0 \qquad \boldsymbol{L}(z)\Psi(z,\zeta_1) = 0 \tag{65}$$

$$\boldsymbol{L}(z)\dot{\Phi}(z,\zeta_1) = 4k_1^2\zeta_1^{-1}\Phi(z,\zeta_1) \qquad \boldsymbol{L}(z)\dot{\Psi}(z,\zeta_1) = -4k_1^2\zeta_1^{-1}\Psi(z,\zeta_1). \tag{66}$$

Here the dots denote derivatives with respect to $\zeta$. Explicit expressions for such $t$-independent Jost solutions and squared Jost solutions for one-soliton can be found in the appendix.

## 4. Closure of the squared Jost solutions

### *4.1. Inner products and orthogonalities*

As in the case of bright solitons [6, 14], it is probable that $\Phi(\zeta)$, $\Phi(\zeta_1)$, $\dot{\Phi}(\zeta_1)$ and their counterparts $\tilde{\Phi}(\zeta)$, $\tilde{\Phi}(\zeta_1)$, $\dot{\tilde{\Phi}}(\zeta_1)$ construct a complete set. If such a probability comes true, any solution of (38) can be expanded with them, i.e.

$$\begin{aligned} f(z) = &-\frac{1}{2\pi}\int_C \mathrm{d}\zeta\ f(\zeta)\Phi(z,\zeta) + f_1\Phi(z,\zeta_1) + g_1\dot{\Phi}(z,\zeta_1) \\ &-\frac{1}{2\pi}\int_{\tilde{C}} \mathrm{d}\zeta\ \tilde{f}(\zeta)\tilde{\Phi}(z,\zeta) + \tilde{f}_1\tilde{\Phi}(z,\zeta_1) + \tilde{g}_1\dot{\tilde{\Phi}}(z,\zeta_1). \end{aligned} \tag{67}$$

Here $C$ is a line on the upper half-plane of $\zeta$, from $-\infty + \mathrm{i}0^+$ to $+\infty + \mathrm{i}0^+$, while $\tilde{C}$ is on the lower half-plane, from $-\infty - \mathrm{i}0^+$ to $+\infty - \mathrm{i}0^+$. Such a choice of integral path comes from the fact that $\Phi$ and $\tilde{\Phi}$ are analytic on the upper- and lower half-plane, respectively. However, equations (20) and (21) yield the reduction relations of the squared Jost solutions:

$$\tilde{\Phi}(z,\rho^2\zeta^{-1}) = -\rho^{-2}\zeta^2\Phi(z,\zeta) \tag{68}$$

$$\tilde{\Phi}(z,\overline{\zeta}_1) = -\rho^{-2}\zeta_1^2\Phi(z,\zeta_1) \tag{69}$$

$$\dot{\tilde{\Phi}}(z,\overline{\zeta}_1) = \rho^{-4}\zeta_1^4\dot{\Phi}(z,\zeta_1) + 2\rho^{-4}\zeta_1^3\Phi(z,\zeta_1). \tag{70}$$

For an arbitrary function $\tilde{f}(\zeta)$, we have

$$\int_{\tilde{C}} \mathrm{d}\zeta\ \tilde{f}(\zeta)\tilde{\Phi}(\zeta) = \int_{\tilde{C}} \mathrm{d}\zeta\ \tilde{f}(\zeta)(-\rho^2\zeta^{-2})\Phi(\rho^2\zeta^{-1}) = -\int_C \mathrm{d}\zeta\ \tilde{f}(\rho^2\zeta^{-1})\Phi(\zeta). \tag{71}$$

Thus, equation (67) should be rewritten as

$$f(z) = -\frac{1}{2\pi}\int_C \mathrm{d}\zeta\ f(\zeta)\Phi(z,\zeta) + f_1\Phi(z,\zeta_1) + g_1\dot{\Phi}(z,\zeta_1) \tag{72}$$

i.e. because of the reduction relations, $\tilde{\Phi}(z,\zeta)$, $\tilde{\Phi}(z,\zeta_1)$ and $\dot{\tilde{\Phi}}(z,\zeta_1)$ are not linearly independent of their counterparts without tildes. Because of the non-vanishing boundary condition, it is very difficult to find adjoint operator of $\boldsymbol{L}(z)$. However, if we directly define the adjoint state of $\Phi(z,\zeta)$ as that for bright solitons

$$\Phi(z,\zeta)^{\mathrm{A}} = \Psi(z,\zeta)^{\mathrm{T}}(\mathrm{i}\sigma_2) = \big(-\psi_2^2(z,\zeta) \quad \psi_1^2(z,\zeta)\big) \tag{73}$$

and introduce the corresponding inner product as

$$\langle\Phi(\zeta')|\Phi(\zeta)\rangle = \int_{-\infty}^{\infty} \Phi(z,\zeta')^{\mathrm{A}}\Phi(z,\zeta)\,\mathrm{d}z \tag{74}$$

we have orthogonality of the continuous spectrum:

$$\langle\Phi(\zeta')|\Phi(\zeta)\rangle = -a(\zeta)^2 2\pi\,(1-\rho^2\zeta^{-2})\delta(\zeta-\zeta'). \tag{75}$$

Such a definition is thus appropriate. From equation (63), we have

$$\langle\Phi|(-\sigma_2 L^{\mathrm{T}}\sigma_2) = -4\kappa(\lambda-\lambda_1)\langle\Phi|. \tag{76}$$

This means the corresponding adjoint operator of $L$ [6] should be

$$L^{\mathrm{A}} = -\sigma_2 L^{\mathrm{T}}\sigma_2. \tag{77}$$

We also find that

$$\langle\Phi(\zeta_1)|\Phi(\zeta_1)\rangle = 0 \tag{78}$$

$$\langle\Phi(\zeta_1)|\dot{\Phi}(\zeta_1)\rangle = -\mathrm{i}(1-\rho^2\zeta_1^{-2})\dot{a}(\zeta_1)^2 \tag{79}$$

$$\langle\dot{\Phi}(\zeta_1)|\Phi(\zeta_1)\rangle = -\mathrm{i}(1-\rho^2\zeta_1^{-2})\dot{a}(\zeta_1)^2 \tag{80}$$

$$\langle\dot{\Phi}(\zeta_1)|\dot{\Phi}(\zeta_1)\rangle = -\mathrm{i}\dot{a}(\zeta_1)\ddot{a}(\zeta_1)(1-\rho^2\zeta_1^{-2}) - \mathrm{i}2\dot{a}(\zeta_1)^2\,\rho^2\zeta_1^{-3}. \tag{81}$$

In what follows we will prove that a complete set can be constructed with $\Phi(z,\zeta)$, $\Phi(z,\zeta_1)$ and $\dot{\Phi}(z,\zeta_1)$.

### 4.2. *Completeness*

In Dirac notation, equation (72) reads

$$|f\rangle = -\frac{1}{2\pi}\int_C \mathrm{d}\zeta\; f(\zeta)|\Phi(\zeta)\rangle + f_1|\Phi(\zeta_1)\rangle + g_1|\dot{\Phi}(\zeta_1)\rangle. \tag{82}$$

With equation (75)–(81), we have

$$f(\zeta) = \frac{1}{a(\zeta)^2(1-\rho^2\zeta^{-2})}\langle\Phi(\zeta)|f\rangle \tag{83}$$

$$g_1 = \mathrm{i}\frac{1}{\dot{a}(\zeta_1)^2(1-\rho^2\zeta_1^{-2})}\langle\Phi(\zeta_1)|f\rangle \tag{84}$$

and

$$f_1 = -\mathrm{i}\left\{\frac{\ddot{a}(\zeta_1)}{\dot{a}(\zeta_1)^3(1-\rho^2\zeta_1^{-2})} + \frac{2\,\rho^2\zeta_1^{-3}}{\dot{a}(\zeta_1)^2(1-\rho^2\zeta_1^{-2})^2}\right\}\langle\Phi(\zeta_1)|f\rangle$$
$$+\mathrm{i}\frac{1}{\dot{a}(\zeta_1)^2(1-\rho^2\zeta_1^{-2})}\langle\dot{\Phi}(\zeta_1)|f\rangle. \tag{85}$$

Substituting the above results back in (82), we have

$$I = -\frac{1}{2\pi}\int_C \mathrm{d}\zeta\ \frac{1}{a(\zeta)^2(1-\rho^2\zeta^{-2})}|\Phi(\zeta)\rangle\langle\Phi(\zeta)| + \mathrm{i}\frac{1}{\dot{a}(\zeta_1)^2(1-\rho^2\zeta_1^{-2})}\left\{|\dot{\Phi}(\zeta_1)\rangle\langle\Phi(\zeta_1)| + |\Phi(\zeta_1)\rangle\langle\dot{\Phi}(\zeta_1)|\right\} - \mathrm{i}\left\{\frac{2\rho^2\zeta_1^{-3}}{\dot{a}(\zeta_1)^2(1-\rho^2\zeta_1^{-2})^2} + \frac{\ddot{a}(\zeta_1)}{\dot{a}(\zeta_1)^3(1-\rho^2\zeta_1^{-2})}\right\}|\Phi(\zeta_1)\rangle\langle\Phi(\zeta_1)| \tag{86}$$

or

$$\delta(x-y) = -\frac{1}{2\pi}\int_C \mathrm{d}\zeta\ \frac{1}{a^2(\zeta)(1-\rho^2\zeta^{-2})}\Phi(x,\zeta)\Psi^{\mathrm{T}}(y,\zeta)(\mathrm{i}\sigma_2) + \mathrm{i}\frac{1}{\dot{a}^2(\zeta_1)(1-\rho^2\zeta_1^{-2})}\left\{\dot{\Phi}(x,\zeta_1)\Psi^{\mathrm{T}}(y,\zeta_1)(\mathrm{i}\sigma_2) + \Phi(x,\zeta_1)\dot{\Psi}^{\mathrm{T}}(y,\zeta_1)(\mathrm{i}\sigma_2)\right\} - \mathrm{i}\left\{\frac{2\rho^2\zeta_1^{-3}}{\dot{a}^2(\zeta_1)(1-\rho^2\zeta_1^{-2})^2} + \frac{\ddot{a}(\zeta_1)}{\dot{a}^3(\zeta_1)(1-\rho^2\zeta_1^{-2})}\right\}\Phi(x,\zeta_1)\Psi^{\mathrm{T}}(y,\zeta_1)(\mathrm{i}\sigma_2). \tag{87}$$

That is, if equation (87) can be proved, completeness holds. As $|\zeta| \to \infty$

$$\frac{1}{a^2(\zeta)}\Phi(x,\zeta)\Phi^{\mathrm{T}}(y,\zeta)(\mathrm{i}\sigma_2) = \tilde{\Psi}(x,\zeta)\Psi^{\mathrm{T}}(y,\zeta)(\mathrm{i}\sigma_2) \to A(x,y,\zeta) \tag{88}$$

where

$$A(x,y,\zeta) = \begin{pmatrix} -1 & -\rho^2\zeta^{-2} \\ \rho^2\zeta^{-2} & \rho^4\zeta^{-4} \end{pmatrix}\mathrm{e}^{-\mathrm{i}2\kappa(x-y)}. \tag{89}$$

It is not difficult to prove

$$-\frac{1}{2\pi}\int_C \mathrm{d}\zeta\ \frac{1}{(1-\rho^2\zeta^{-2})}A(x,y,\zeta) = \delta(x-y). \tag{90}$$

Let

$$-\frac{1}{2\pi}\int_C \mathrm{d}\zeta\ \frac{1}{a^2(\zeta)(1-\rho^2\zeta^{-2})}\Phi(x,\zeta)\Psi^{\mathrm{T}}(y,\zeta) = J(x,y) + \delta(x-y) \tag{91}$$

where

$$J(x,y) = -\frac{1}{2\pi}\int_C \mathrm{d}\zeta\ \frac{1}{a^2(\zeta)(1-\rho^2\zeta^{-2})}\{\Phi(x,\zeta)\Psi^{\mathrm{T}}(y,\zeta)(\mathrm{i}\sigma_2) - a^2(\zeta)A(x,y,\zeta)\}. \tag{92}$$

The asymptotic behaviour of the integrand of $J(x,y)$ on $|\zeta| \to \infty$ is

$$\mathrm{O}(|\zeta|^{-2})\mathrm{e}^{-\mathrm{i}2\kappa(x-y)}. \tag{93}$$

When $x < y$, according to Jordan's lemma, integration of the integrand of $J(x,y)$ on a sufficiently large semicircle on the upper half-plane of $\zeta$ is zero. Even when $x = y$, the asymptotic behaviour of $\mathrm{O}(|\zeta|^{-2})$ also ensures such an integration to be zero. Hence, as

$x \leqslant y$, a semicircle can be added to construct a closed contour integration in (92). In such a contour, there is only one second-order pole $\zeta = \zeta_1$ contributed by $a^2(\zeta)$:

$$\begin{aligned}
J(x,y)|_{x\leqslant y} &= -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left\{\frac{(\zeta-\zeta_1)^2}{a^2(\zeta)(1-\rho^2\zeta^{-2})}[\Phi(x,\zeta)\Psi^{\mathrm{T}}(y,\zeta)(\mathrm{i}\sigma_2) - a^2(\zeta)A(x,y,\zeta)]\right\}\bigg|_{\zeta=\zeta_1} \\
&= -\mathrm{i}\frac{1}{\dot{a}^2(\zeta_1)}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left\{\frac{1}{1-\rho^2\zeta^{-2}}\Phi(x,\zeta)\Psi^{\mathrm{T}}(y,\zeta)(\mathrm{i}\sigma_2)\right\}\bigg|_{\zeta=\zeta_1} \\
&\quad + \mathrm{i}\frac{\ddot{a}(\zeta_1)}{\dot{a}^3(\zeta_1)}\left\{\frac{1}{1-\rho^2\zeta^{-2}}\Phi(x,\zeta_1)\Psi^{\mathrm{T}}(y,\zeta_1)(\mathrm{i}\sigma_2)\right\} \\
&= -\mathrm{i}\frac{1}{\dot{a}^2(\zeta_1)(1-\rho^2\zeta_1^{-2})}\{\dot{\Phi}(x,\zeta_1)\Psi^{\mathrm{T}}(y,\zeta_1)(\mathrm{i}\sigma_2) + \Phi(x,\zeta_1)\dot{\Psi}^{\mathrm{T}}(y,\zeta_1)(\mathrm{i}\sigma_2)\} \\
&\quad + \mathrm{i}\left\{\frac{2\rho^2\zeta_1^{-3}}{\dot{a}^2(\zeta_1)(1-\rho^2\zeta_1^{-2})^2} + \frac{\ddot{a}(\zeta_1)}{\dot{a}^3(\zeta_1)(1-\rho^2\zeta_1^{-2})}\right\}\Phi(x,\zeta_1)\Psi^{\mathrm{T}}(y,\zeta_1)(\mathrm{i}\sigma_2).
\end{aligned} \tag{94}$$

Then equation (87), i.e. completeness, is proved to be valid for the case $x \leqslant y$. As $x > y$, according to Jordan's lemma, we have to add a sufficiently large semicircle on the lower half-plane to construct a closed contour in (92). Now the contour contains a pair of simple poles $\zeta = \pm\rho$ on the real axis. In the reflectionless case, equation (26) yields $\Phi(x,\zeta) = a^2(\zeta)\tilde{\Psi}(x,\zeta)$ and $\Psi(x,\zeta) = a^2(\zeta)\tilde{\Phi}(x,\zeta)$. Then

$$\begin{aligned}
J(x,y)|_{x>y} &= -\frac{1}{2\pi}\int_C \mathrm{d}\zeta\,\frac{1}{\tilde{a}^2(\zeta)(1-\rho^2\zeta^{-2})}\{\tilde{\Psi}(x,\zeta)\tilde{\Phi}^{\mathrm{T}}(y,\zeta)(\mathrm{i}\sigma_2) - \tilde{a}^2(\zeta)A(x,y,\zeta)\} \\
&= \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left\{\frac{(\zeta-\overline{\zeta}_1)^2}{\tilde{a}^2(\zeta)(1-\rho^2\zeta^{-2})}[\tilde{\Psi}(x,\zeta)\tilde{\Phi}^{\mathrm{T}}(y,\zeta)(\mathrm{i}\sigma_2) - \tilde{a}^2(\zeta)A(x,y,\zeta)]\right\}\bigg|_{\zeta=\overline{\zeta}_1} \\
&\quad + \mathrm{i}\frac{\rho}{2\tilde{a}^2(\rho)}\tilde{\Psi}(x,\rho)\tilde{\Phi}^{\mathrm{T}}(y,\rho)(\mathrm{i}\sigma_2) - \mathrm{i}\frac{\rho}{2\tilde{a}^2(-\rho)}\tilde{\Psi}(x,-\rho)\tilde{\Phi}^{\mathrm{T}}(y,-\rho)(\mathrm{i}\sigma_2) \\
&= \mathrm{i}\frac{1}{\dot{\tilde{a}}^2(\overline{\zeta}_1)}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left\{\frac{1}{1-\rho^2\zeta^{-2}}\tilde{\Psi}(x,\zeta)\tilde{\Phi}^{\mathrm{T}}(y,\zeta)(\mathrm{i}\sigma_2)\right\}\bigg|_{\zeta=\overline{\zeta}_1} \\
&\quad + \mathrm{i}\frac{\ddot{\tilde{a}}(\overline{\zeta}_1)}{\dot{\tilde{a}}^3(\overline{\zeta}_1)}\left\{\frac{1}{1-\rho^2\overline{\zeta}_1^{-2}}\tilde{\Psi}(x,\zeta_1)\tilde{\Phi}^{\mathrm{T}}(y,\zeta_1)(\mathrm{i}\sigma_2)\right\} \\
&\quad + \mathrm{i}\frac{\rho}{2}\tilde{\Psi}(x,\rho)\tilde{\Phi}^{\mathrm{T}}(y,\rho)(\mathrm{i}\sigma_2) - \mathrm{i}\frac{\rho}{2}\tilde{\Psi}(x,-\rho)\tilde{\Phi}^{\mathrm{T}}(y,-\rho)(\mathrm{i}\sigma_2) \\
&= -\mathrm{i}\frac{1}{\dot{\tilde{a}}^2(\overline{\zeta}_1)(1-\rho^2\overline{\zeta}_1^{-2})}\{\dot{\tilde{\Psi}}(x,\zeta_1)\tilde{\Phi}^{\mathrm{T}}(y,\zeta_1)(\mathrm{i}\sigma_2) + \tilde{\Psi}(x,\zeta_1)\dot{\tilde{\Phi}}^{\mathrm{T}}(y,\zeta_1)(\mathrm{i}\sigma_2)\} \\
&\quad + \mathrm{i}\left\{\frac{2\rho^2\overline{\zeta}_1^{-3}}{\dot{\tilde{a}}^2(\overline{\zeta}_1)(1-\rho^2\overline{\zeta}_1^{-2})^2} + \frac{\ddot{\tilde{a}}(\overline{\zeta}_1)}{\dot{\tilde{a}}^3(\overline{\zeta}_1)(1-\rho^2\overline{\zeta}_1^{-2})}\right\}\tilde{\Psi}(x,\overline{\zeta}_1)\tilde{\Phi}^{\mathrm{T}}(y,\overline{\zeta}_1)(\mathrm{i}\sigma_2) \\
&\quad + \mathrm{i}\frac{\rho}{2}\Phi(x,\rho)\Psi^{\mathrm{T}}(y,\rho)(\mathrm{i}\sigma_2) - \mathrm{i}\frac{\rho}{2}\Phi(x,-\rho)\Psi^{\mathrm{T}}(y,-\rho)(\mathrm{i}\sigma_2).
\end{aligned} \tag{95}$$

Substituting the explicit form of the Jost solutions, we find

$$-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left\{\frac{(\zeta-\zeta_1)^2}{a^2(\zeta)(1-\rho^2\zeta^{-2})}\Phi(x,\zeta)\Psi^{\mathrm{T}}(y,\zeta)(\mathrm{i}\sigma_2)\right\}\Bigg|_{\zeta=\zeta_1}$$

$$-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left\{\frac{(\zeta-\overline{\zeta}_1)^2}{\tilde{a}^2(\zeta)(1-\rho^2\zeta^{-2})}\tilde{\Psi}(x,\zeta)\tilde{\Phi}^{\mathrm{T}}(y,\zeta)(\mathrm{i}\sigma_2)\right\}\Bigg|_{\zeta=\overline{\zeta}_1}$$

$$=\mathrm{i}\frac{\rho}{2}\Phi(x,\rho)\Psi^{\mathrm{T}}(y,\rho)(\mathrm{i}\sigma_2)-\mathrm{i}\frac{\rho}{2}\Phi(x,-\rho)\Psi^{\mathrm{T}}(y,-\rho)(\mathrm{i}\sigma_2)$$

$$=\begin{pmatrix} F & G \\ \overline{G} & \overline{F}\end{pmatrix} \tag{96}$$

where

$$F=\mathrm{i}\frac{\lambda_1}{2}\{\mathrm{sech}^2(k_1z)+\mathrm{sech}^2(k_1y)-\mathrm{sech}^2(k_1z)\,\mathrm{sech}^2(k_1y)\}$$

$$+\frac{k}{2}\{\mathrm{sech}^2(k_1z)\tanh(k_1y)-\mathrm{sech}^2(k_1y)\tanh(k_1z)\} \tag{97}$$

$$G=\mathrm{i}\frac{\lambda_1}{4}\mathrm{e}^{-\mathrm{i}2\beta_1}\,\mathrm{sech}^2(k_1z)\,\mathrm{sech}^2(k_1y)\{\cosh(2k_1z)+\cosh(2k_1y)\}$$

$$-\frac{k_1}{4}\mathrm{e}^{-\mathrm{i}2\beta_1}\,\mathrm{sech}^2(k_1z)\,\mathrm{sech}^2(k_1y)\{\sinh(2k_1z)+\sinh(2k_1y)\}. \tag{98}$$

That is, $J(x,y)|_{x\leqslant y}=J(x,y)|_{x>y}$, i.e. completeness of the squared Jost solution is proved for the one-soliton case.

## 5. Expansion in the complete set of squared Jost solutions

Under the transformation (52) equation (38) becomes

$$(\mathrm{i}\partial_t-\boldsymbol{L}(z))|q\rangle=|\boldsymbol{R}\rangle. \tag{99}$$

$|\boldsymbol{q}\rangle$ can be expanded in the obtained complete set

$$|\boldsymbol{q}\rangle=-\frac{1}{2\pi}\int_C\mathrm{d}\zeta\;q(\zeta)|\Phi(\zeta)\rangle+q_1|\Phi(\zeta_1)\rangle+q_2|\dot{\Phi}(\zeta_1)\rangle. \tag{100}$$

With equations (61), (65) and (66), one can find

$$(\mathrm{i}\partial_t-\boldsymbol{L}(z))|\boldsymbol{q}\rangle=-\frac{1}{2\pi}\int_C\mathrm{d}\zeta\;[iq_t(\zeta)+4\kappa(\lambda-\lambda_1)q(\zeta)]|\Phi(\zeta)\rangle$$

$$+[\mathrm{i}q_{1t}-4k_1^2\zeta_1^{-1}q_2]|\Phi(\zeta_1)\rangle+\mathrm{i}q_{2t}|\dot{\Phi}(\zeta_1)\rangle. \tag{101}$$

Let $\langle\Phi(\zeta_1)|$, $\langle\dot{\Phi}(\zeta_1)|$ and $\langle\Phi(\zeta)|$ act on both sides of (99), respectively; then using the inner products (75)–(81), we have

$$\mathrm{i}q_{2t}\langle\Phi(\zeta_1)|\dot{\Phi}(\zeta_1)\rangle=\langle\Phi(\zeta_1)|\boldsymbol{R}\rangle \tag{102}$$

$$(\mathrm{i}q_{1t}-4k_1\zeta_1^{-1}q_2)\langle\dot{\Phi}(\zeta_1)|\Phi(\zeta_1)\rangle+\mathrm{i}q_{2t}\langle\dot{\Phi}(\zeta_1)|\Phi(\zeta_1)\rangle=\langle\dot{\Phi}(\zeta_1)|\boldsymbol{R}\rangle \tag{103}$$

$$\mathrm{i}q_t(\zeta)+4\kappa(\lambda-\lambda_1)q(\zeta)=\frac{\zeta}{2a^2(\zeta)}\kappa^{-1}\langle\Phi(\zeta)|\boldsymbol{R}\rangle. \tag{104}$$

For the one-soliton case the effective $R$ is independent of $t$ in the TFR, the above equations can be easily solved without introduction of Green's function. From equation (102) with the initial condition $q_2|_{t=0} = 0$, we have

$$q_2 = -\mathrm{i}\frac{\langle \Phi(\zeta_1)|\boldsymbol{R}\rangle}{\langle \Phi(\zeta_1)|\dot{\Phi}(\zeta_1)\rangle} t. \tag{105}$$

This is a so-called secular term which infinitely enlarges with $t$. The condition to diminish this term

$$\langle \Phi(\zeta_1)|\boldsymbol{R}\rangle = \int_{-\infty}^{+\infty} \mathrm{d}z\ \Psi^{\mathrm{T}}(z,\zeta_1)(\mathrm{i}\sigma_2)\boldsymbol{R} = 0 \tag{106}$$

is called a secular condition. Substituting equation (106) in (103), we have

$$q_1 = -\mathrm{i}\frac{\langle \dot{\Phi}(\zeta_1)|\boldsymbol{R}\rangle}{\langle \dot{\Phi}(\zeta_1)|\Phi(\zeta_1)\rangle} t \tag{107}$$

which is another secular term, and the corresponding secular condition

$$\langle \dot{\Phi}(\zeta_1)|\boldsymbol{R}\rangle = \int_{-\infty}^{+\infty} \mathrm{d}z\ \dot{\Psi}^{\mathrm{T}}(z,\zeta_1)(\mathrm{i}\sigma_2)\boldsymbol{R} = 0 \tag{108}$$

is obtained. Taking into account (106), this secular condition can be simplified as

$$\langle \Omega(\zeta_1)|\boldsymbol{R}\rangle = \int_{-\infty}^{+\infty} \mathrm{d}z\ \Omega^{\mathrm{A}}(z,\zeta_1)\boldsymbol{R} = 0 \tag{109}$$

where

$$\Omega(z,\zeta_1)^{\mathrm{A}} = \left[\dot{\Psi}^{\mathrm{T}}(\zeta_1) - \mathrm{i}\frac{1}{k_1}\Psi^{\mathrm{T}}(z,\zeta_1)\right](\mathrm{i}\sigma_2) \tag{110}$$

has a simpler explicit expression (see the appendix).

## 6. The zeroth-order approximation and the adiabatic solution

Upon substitution of the explicit expression for the squared Jost solutions (see the appendix), the secular conditions (106) and (109) become

$$\mathrm{i}2k_1\mathrm{e}^{\mathrm{i}\beta_1}\langle \Phi(\zeta_1)|\boldsymbol{R}\rangle = \int_{-\infty}^{+\infty} \mathrm{d}\theta_1\ \operatorname{sech}^2\theta_1 \operatorname{Im}\{\mathrm{e}^{\mathrm{i}\beta_1}R\} = 0 \tag{111}$$

$$\begin{aligned} k_1^2\mathrm{e}^{\mathrm{i}2\beta_1}\langle \Omega(\zeta_1)|\boldsymbol{R}\rangle = {}& \frac{\lambda_1}{\rho}\int_{-\infty}^{+\infty} \mathrm{d}\theta_1\ \theta_1 \operatorname{sech}^2\theta_1 \operatorname{Im}[\mathrm{e}^{\mathrm{i}\beta_1}R] \\ & - \int_{-\infty}^{+\infty} \mathrm{d}\theta_1\ \operatorname{sech}\theta_1 \mathrm{e}^{-\theta_1} \operatorname{Im}[\mathrm{e}^{\mathrm{i}2\beta_1}R] = 0. \end{aligned} \tag{112}$$

The secular conditions (106) and (109) can also be written as

$$\langle \Phi(\zeta_1)|\mathrm{i}\boldsymbol{u}_1'\rangle = \langle \Phi(\zeta_1)|\boldsymbol{r}\rangle \tag{113}$$

and

$$\langle \Omega(\zeta_1)|\mathrm{i}\boldsymbol{u}_1'\rangle = \langle \Omega(\zeta_1)|\boldsymbol{r}\rangle. \tag{114}$$

Since the free parameter $x_1$, which should have been slowly time dependent has been absorbed in $z$, a parameter $z_c$ must be introduced to consider the shift of the soliton centre, i.e. $\theta_1$ in (4) should be written as

$$\theta_1 = k_1(z - z_c) \tag{115}$$

where $z_c$ satisfies the initial condition

$$z_c|_{t=0} = 0. \tag{116}$$

With the following expressions:

$$\theta_1' = \frac{k_1'}{k_1}\theta_1 - k_1 z_c' \tag{117}$$

$$e^{i\beta_1}u_1 = \lambda_1 + ik_1 \tanh\theta_1 \tag{118}$$

$$e^{i2\beta_1}u_1 = \rho - \frac{k_1^2}{\rho}\operatorname{sech}\theta_1 e^{\theta_1} + i\frac{k_1\lambda_1}{\rho}\operatorname{sech}\theta_1 e^{\theta_1} \tag{119}$$

$$\begin{aligned}\operatorname{Im}[e^{i\beta_1} i u_1'] &= \operatorname{Im}[i(e^{i\beta_1}u_1)' + \beta_1'(e^{i\beta_1}u_1)] \\ &= \operatorname{Re}[(e^{i\beta_1}u_1)'] + \beta_1' \operatorname{Im}[e^{i\beta_1}u_1] = \lambda_1' + k_1\tanh\theta_1\beta_1'\end{aligned} \tag{120}$$

$$\begin{aligned}\operatorname{Im}[e^{i2\beta_1} i u_1'] &= \operatorname{Im}[i(e^{i2\beta_1}u_1)' + 2\beta_1'(e^{i2\beta_1}u_1)] \\ &= \operatorname{Re}[(e^{i2\beta_1}u_1)'] + 2\beta_1' \operatorname{Im}[e^{i2\beta_1}u_1] \\ &= \rho' - \frac{k_1}{\rho}k_1'\theta_1\operatorname{sech}^2\theta_1 + \frac{k_1^3}{\rho}z_c'\operatorname{sech}^2\theta_1 - \frac{k_1^2}{\rho^2}\rho'\operatorname{sech}\theta_1 e^{\theta_1}\end{aligned} \tag{121}$$

we find that

$$2ik_1 e^{i\beta_1}\langle\Phi(\zeta_1)|i\boldsymbol{u}_1'\rangle = 2\lambda_1' \tag{122}$$

and

$$k_1^2 e^{i2\beta_1}\langle\Omega(\zeta_1)|i\boldsymbol{u}_1'\rangle = -2\mathcal{L}k_1\rho' + \frac{k_1^2}{\rho^2}\rho' - 2\frac{k_1^3}{\rho}z_c' \tag{123}$$

where $2\mathcal{L}$ is the length of the system with $2\mathcal{L} \to \infty$. The $2\mathcal{L}$ form divergence comes from the fact that dark solitons have infinite background energy. Then we have

$$\epsilon\lambda' = \frac{1}{2}\int_{-\infty}^{+\infty} d\theta_1\ \operatorname{sech}^2\theta_1 \operatorname{Im}\{e^{i\beta_1}\epsilon r\} \tag{124}$$

and

$$\begin{aligned}-2\mathcal{L}k_1\epsilon\rho' + \frac{k_1^2}{\rho^2}\epsilon\rho' - 2\frac{k_1^3}{\rho}\epsilon z_c' = \frac{\lambda_1}{\rho}\int_{-\infty}^{+\infty} d\theta_1\ \theta_1\operatorname{sech}^2\theta_1\operatorname{Im}[e^{i\beta_1}\epsilon r] \\ - \int_{-\infty}^{+\infty} d\theta_1\ \operatorname{sech}\theta_1 e^{-\theta_1}\operatorname{Im}[e^{i2\beta_1}\epsilon r].\end{aligned} \tag{125}$$

The secular condition (125) consists of two independent equations: the term diverging in the form of $2\mathcal{L}$ and the finite term in (125) must equal zero separately. We have got three independent equations to determine the evolution of all soliton parameters because only two out of $\lambda_1$, $k_1$, $\beta_1$ and $\rho$ are independent. For vanishing perturbations ($r \to 0$, as $|x| \to \infty$), there is no $2\mathcal{L}$ term on the right-hand side of (125), then $\rho' = 0$. On the other hand, for non-vanishing perturbations($r \to$ constant, as $|x| \to \infty$), $2\mathcal{L}$ terms will appear in the second integral on the right-hand side of (125), $\rho' \neq 0$.

## 7. First-order correction

Now $\mathrm{i}u_1'$ has been determined by the secular conditions, and so has the effective source $R$. Then $q(\zeta)$ can be easily found from

$$\mathrm{i}q_t(\zeta) + 4\kappa(\lambda - \lambda_1)q(\zeta) = \frac{\zeta}{2a^2(\zeta)}\kappa^{-1}\langle\Phi(\zeta)|\boldsymbol{R}\rangle \qquad q|_{t=0} = 0 \tag{126}$$

i.e.

$$\epsilon q(\zeta, t) = \frac{\zeta}{8\kappa^2 a^2(\zeta)(\lambda - \lambda_1)}[1 - \mathrm{e}^{\mathrm{i}4\kappa(\lambda-\lambda_1)t}]\langle\Phi(\zeta)|\epsilon\boldsymbol{R}\rangle \tag{127}$$

with

$$\begin{aligned}
\langle\Phi(\zeta)|\epsilon\boldsymbol{R}\rangle &= \int_{-\infty}^{+\infty} \mathrm{d}z\ \Phi(z,\zeta)^{\mathrm{A}}\epsilon\boldsymbol{R} \\
&= -\int_{-\infty}^{+\infty} \mathrm{d}z\ [\psi_2^2(z,\zeta)\epsilon R + \psi_1^2(z,\zeta)\epsilon\overline{R}] \\
&= -\frac{1}{k_1}\int_{-\infty}^{+\infty} \mathrm{d}\theta_1\ \exp\left(\mathrm{i}2\frac{\kappa}{k_1}\theta_1\right)\epsilon(R - \rho^2\zeta^{-2}\overline{R}) \\
&\quad + \mathrm{i}\frac{2}{\zeta - \overline{\zeta}_1}\int_{-\infty}^{+\infty} \mathrm{d}\theta_1\ \exp\left(\mathrm{i}2\frac{\kappa}{k_1}\theta_1\right)\operatorname{sech}\theta_1 \mathrm{e}^{-\theta_1}\epsilon(R - \rho\zeta^{-1}\mathrm{e}^{-\mathrm{i}\beta_1}\overline{R}) \\
&\quad + \mathrm{i}\frac{2k_1\mathrm{e}^{-\mathrm{i}\beta_1}}{(\zeta - \overline{\zeta}_1)^2}\int_{-\infty}^{+\infty} \mathrm{d}\theta_1\ \exp\left(\mathrm{i}2\frac{\kappa}{k_1}\theta_1\right)\operatorname{sech}^2\theta_1 \mathrm{e}^{-2\theta_1}\operatorname{Im}[\mathrm{e}^{\mathrm{i}\beta_1}\epsilon R].
\end{aligned} \tag{128}$$

Thus the first-order correction can be obtained from

$$\epsilon q(z, t) = -\frac{1}{2\pi}\int_C \mathrm{d}\zeta\ \epsilon q(\zeta, t)\phi_1^2(z, \zeta). \tag{129}$$

Usually the integrand of (129) is very complex and is impossible to calculate exactly.

## 8. Example: a dark soliton evolution under damping

When a dark soliton is affected by damping

$$\epsilon r[u] = -\mathrm{i}\gamma u_1 \tag{130}$$

it is obvious that $r[u]$ is non-vanishing and $\rho$ must evolve with time. By employing the formulae provided above, one can find

$$\epsilon r[u] = -\mathrm{i}\gamma u_1 \qquad \operatorname{Im}[\mathrm{e}^{\mathrm{i}\beta_1}\epsilon r] = -\gamma\lambda_1 \tag{131}$$

$$\operatorname{Im}[\mathrm{e}^{\mathrm{i}2\beta_1}\epsilon r] = -\gamma\rho + \gamma\frac{k_1^2}{\rho}\operatorname{sech}\theta_1 \mathrm{e}^{\theta_1} \tag{132}$$

$$\epsilon\lambda_1' = \frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}\theta_1\ \operatorname{sech}^2\theta_1(-\gamma\lambda_1) = -\gamma\lambda_1 \tag{133}$$

and

$$-2\mathcal{L}k_1\epsilon\rho' + \frac{k_1^2}{\rho^2}\epsilon\rho' - 2\frac{k_1^3}{\rho}\epsilon z_c'$$

$$= -\gamma\frac{\lambda_1^2}{\rho}\int_{-\infty}^{+\infty} d\theta_1\ \theta_1 \operatorname{sech}^2\theta_1 + \gamma\rho\int_{-\infty}^{+\infty} d\theta_1\ \operatorname{sech}\theta_1 e^{-\theta_1}$$

$$-\gamma\frac{k_1^2}{\rho}\int_{-\infty}^{+\infty} d\theta_1\ \operatorname{sech}^2\theta_1$$

$$= 2\mathcal{L}\gamma\rho - 2\gamma\frac{k_1^2}{\rho}. \tag{134}$$

Then

$$\epsilon\lambda_1' = -\gamma\lambda_1 \tag{135}$$

$$\epsilon\rho' = -\gamma\rho \tag{136}$$

$$-k_1\epsilon z_c' = -\tfrac{1}{2}\gamma. \tag{137}$$

These equations yield

$$\lambda_1(t) = \lambda_1(0)e^{-\gamma t} \tag{138}$$

$$\rho(t) = \rho(0)e^{-\gamma t} \tag{139}$$

$$k_1(t) = k_1(0)e^{-\gamma t} \tag{140}$$

$$\beta_1(t) = \beta_1(0) \tag{141}$$

$$z_c(t) = \frac{1}{2k_1(0)}\gamma t. \tag{142}$$

It can be verified that the so-called adiabatic method [18, 19] can also yield the same results as above, except that for $z_c$.

Thus the effective source is determined to be

$$\epsilon R = \epsilon(r - iu_1') = -\gamma e^{-i\beta_1}k_1(\theta_1 + \tfrac{1}{2})\operatorname{sech}^2\theta_1. \tag{143}$$

We have

$$\operatorname{Im}[e^{i\beta_1}\epsilon R] = 0 \tag{144}$$

$$\epsilon(R - \rho^2\zeta^{-2}\overline{R}) = -\gamma k_1 e^{-i\beta_1}\zeta^{-2}(\zeta^2 - \zeta_1^2)(\theta_1 + \tfrac{1}{2})\operatorname{sech}^2\theta_1 \tag{145}$$

$$\epsilon(R - \rho\zeta^{-1}e^{-i\beta_1}\overline{R}) = -\gamma k_1 e^{-i\beta_1}\zeta^{-1}(\zeta - \zeta_1)(\theta_1 + \tfrac{1}{2})\operatorname{sech}^2\theta_1 \tag{146}$$

$$\langle\Phi(\zeta)|\epsilon\boldsymbol{R}\rangle = -\gamma e^{-i\beta_1}\zeta^{-2}(\zeta^2 - \zeta_1^2)\int_{-\infty}^{+\infty} d\theta_1\ \exp\left(i2\frac{\kappa}{k_1}\theta_1\right)\left(\theta_1 + \frac{1}{2}\right)\operatorname{sech}^2\theta_1$$

$$+ i\gamma 2k_1 e^{-i\beta_1}\zeta^{-1}\frac{\zeta - \zeta_1}{\zeta - \overline{\zeta}_1}\int_{-\infty}^{+\infty} d\theta_1\ \exp\left(i2\frac{\kappa}{k_1}\theta_1\right)\left(\theta_1 + \frac{1}{2}\right)\operatorname{sech}^3\theta_1 e^{-\theta_1}$$

$$= \gamma\pi e^{-i\beta_1}\zeta^{-2}(\zeta^2 - \zeta_1^2)\operatorname{cosech}\left(\pi\frac{\kappa}{k_1}\right)\left\{\left(\frac{\kappa}{k_1} - i\right) + i\pi\frac{\kappa}{k_1}\coth\left(\pi\frac{\kappa}{k_1}\right)\right\}$$

$$-\gamma\mathrm{i}2\pi k_1\mathrm{e}^{-\mathrm{i}\beta_1}\zeta^{-1}\frac{\zeta-\zeta_1}{\zeta-\bar{\zeta}_1}\operatorname{cosech}\left(\pi\frac{\kappa}{k_1}\right)$$
$$\times\left\{\mathrm{i}\frac{\kappa^2}{k_1^2}+\frac{\kappa}{k_1}+\mathrm{i}-\pi\frac{\kappa}{k_1}\left(\frac{\kappa}{k_1}+\mathrm{i}\right)\coth\left(\pi\frac{\kappa}{k_1}\right)\right\}. \tag{147}$$

Then the first-order correction $q$ can be discussed with (129).

## 9. Conclusion

A direct perturbation approach for the dark one-soliton case is developed in this paper, which is founded on a rigorous proof of the completeness of the squared Jost solutions. A general procedure for adiabatic solutions is given by providing the evolution of all one-soliton parameters. Difficulties caused by the infinite background energy are overcome. A formula for calculating the first-order correction is also obtained.

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## Appendix. Explicit expressions of Jost solutions and squared Jost solutions for one-soliton case

$$\phi_1(x,t,\zeta)=\frac{\zeta-\zeta_1}{\zeta-\bar{\zeta}_1}\mathrm{e}^{-\mathrm{i}\beta_1}\mathrm{e}^{-\mathrm{i}\kappa x}+\mathrm{i}\frac{1}{\zeta-\bar{\zeta}_1}\mathrm{e}^{-\mathrm{i}\beta_1}k_1\operatorname{sech}\theta_1\mathrm{e}^{-\theta_1}\mathrm{e}^{-\mathrm{i}\kappa x} \tag{A.1}$$

$$\phi_2(x,t,\zeta)=\mathrm{i}\rho\zeta^{-1}\frac{\zeta-\zeta_1}{\zeta-\bar{\zeta}_1}\mathrm{e}^{-\mathrm{i}\beta_1}\mathrm{e}^{-\mathrm{i}\kappa x}-\frac{1}{\zeta-\bar{\zeta}_1}k_1\operatorname{sech}\theta_1\mathrm{e}^{-\theta_1}\mathrm{e}^{-\mathrm{i}\kappa x} \tag{A.2}$$

$$\psi_1(x,t,\zeta)=-\mathrm{i}\rho\zeta^{-1}\mathrm{e}^{\mathrm{i}\kappa x}-\frac{1}{\zeta-\bar{\zeta}_1}\mathrm{e}^{-\mathrm{i}\beta_1}k_1\operatorname{sech}\theta_1\mathrm{e}^{-\theta_1}\mathrm{e}^{\mathrm{i}\kappa x} \tag{A.3}$$

$$\psi_2(x,t,\zeta)=\mathrm{e}^{\mathrm{i}\kappa x}-\mathrm{i}\frac{1}{\zeta-\bar{\zeta}_1}k_1\operatorname{sech}\theta_1\mathrm{e}^{-\theta_1}\mathrm{e}^{\mathrm{i}\kappa x}. \tag{A.4}$$

$$\phi_1(z,\zeta)=\frac{\zeta-\zeta_1}{\zeta-\bar{\zeta}_1}\mathrm{e}^{-\mathrm{i}\beta_1}\mathrm{e}^{-\mathrm{i}\kappa z}+\mathrm{i}\frac{1}{\zeta-\bar{\zeta}_1}\mathrm{e}^{-\mathrm{i}\beta_1}k_1\operatorname{sech}\theta_1\mathrm{e}^{-\theta_1}\mathrm{e}^{-\mathrm{i}\kappa z} \tag{A.5}$$

$$\phi_2(z,\zeta)=\mathrm{i}\rho\zeta^{-1}\frac{\zeta-\zeta_1}{\zeta-\bar{\zeta}_1}\mathrm{e}^{-\mathrm{i}\beta_1}\mathrm{e}^{-\mathrm{i}\kappa z}-\frac{1}{\zeta-\bar{\zeta}_1}k_1\operatorname{sech}\theta_1\mathrm{e}^{-\theta_1}\mathrm{e}^{-\mathrm{i}\kappa z} \tag{A.6}$$

$$\psi_1(z,\zeta)=-\mathrm{i}\rho\zeta^{-1}\mathrm{e}^{\mathrm{i}\kappa z}-\frac{1}{\zeta-\bar{\zeta}_1}\mathrm{e}^{-\mathrm{i}\beta_1}k_1\operatorname{sech}\theta_1\mathrm{e}^{-\theta_1}\mathrm{e}^{\mathrm{i}\kappa z} \tag{A.7}$$

$$\psi_2(z,\zeta)=\mathrm{e}^{\mathrm{i}\kappa z}-\mathrm{i}\frac{1}{\zeta-\bar{\zeta}_1}k_1\operatorname{sech}\theta_1\mathrm{e}^{-\theta_1}\mathrm{e}^{\mathrm{i}\kappa z} \tag{A.8}$$

$$\phi_1(z,\zeta_1)=\tfrac{1}{2}\mathrm{e}^{-\mathrm{i}\beta_1}\operatorname{sech}\theta_1 \tag{A.9}$$

$$\phi_2(z,\zeta_1)=\tfrac{1}{2}\mathrm{i}\operatorname{sech}\theta_1 \tag{A.10}$$

$$\psi_1(z,\zeta_1) = -\tfrac{1}{2}\mathrm{i}\mathrm{e}^{-\mathrm{i}\beta_1}\operatorname{sech}\theta_1 \tag{A.11}$$

$$\psi_2(z,\zeta_1) = \tfrac{1}{2}\operatorname{sech}\theta_1 \tag{A.12}$$

$$\dot\phi_1(z,\zeta_1) = -\mathrm{i}\frac{\lambda_1}{2k_1\rho}\mathrm{e}^{-\mathrm{i}2\beta_1}\theta_1\operatorname{sech}\theta_1 + \mathrm{i}\frac{1}{2k_1}\phi_1(z,\zeta_1) - \mathrm{i}\frac{1}{2k_1}\mathrm{e}^{-\mathrm{i}\beta_1}\mathrm{e}^{\theta_1} \tag{A.13}$$

$$\dot\phi_2(z,\zeta_1) = \frac{\lambda_1}{2k_1\rho}\mathrm{e}^{-\mathrm{i}\beta_1}\theta_1\operatorname{sech}\theta_1 + \mathrm{i}\frac{1}{2k_1}\phi_2(z,\zeta_1) + \frac{1}{2k_1}\mathrm{e}^{-\mathrm{i}2\beta_1}\mathrm{e}^{\theta_1} \tag{A.14}$$

$$\dot\psi_1(z,\zeta_1) = \frac{\lambda_1}{2k_1\rho}\mathrm{e}^{-\mathrm{i}2\beta_1}\theta_1\operatorname{sech}\theta_1 + \mathrm{i}\frac{1}{2k_1}\psi_1(z,\zeta_1) - \frac{1}{2k_1}\mathrm{e}^{-\mathrm{i}3\beta_1}\mathrm{e}^{-\theta_1} \tag{A.15}$$

$$\dot\psi_2(z,\zeta_1) = \mathrm{i}\frac{\lambda_1}{2k_1\rho}\mathrm{e}^{-\mathrm{i}\beta_1}\theta_1\operatorname{sech}\theta_1 + \mathrm{i}\frac{1}{2k_1}\psi_2(z,\zeta_1) - \mathrm{i}\frac{1}{2k_1}\mathrm{e}^{-\theta_1} \tag{A.16}$$

$$\Phi(z,\zeta) = \begin{pmatrix} \{(\zeta-\zeta_1) + \mathrm{i}k_1\operatorname{sech}\theta_1\mathrm{e}^{-\theta_1}\}^2 \\ \{\mathrm{i}\rho\zeta^{-1}(\zeta-\zeta_1) - \mathrm{e}^{\mathrm{i}\beta_1}k_1\operatorname{sech}\theta_1\mathrm{e}^{-\theta_1}\}^2 \end{pmatrix} \cdot \frac{\mathrm{e}^{-\mathrm{i}2\beta_1}\mathrm{e}^{-\mathrm{i}2\kappa z}}{(\zeta-\overline{\zeta}_1)^2} \tag{A.17}$$

$$\Psi(z,\zeta) = \begin{pmatrix} \{\mathrm{i}\rho\zeta^{-1}(\zeta-\overline{\zeta}_1) + \mathrm{e}^{-\mathrm{i}\beta_1}k_1\operatorname{sech}\theta_1\mathrm{e}^{-\theta_1}\}^2 \\ \{(\zeta-\overline{\zeta}_1) - \mathrm{i}k_1\operatorname{sech}\theta_1\mathrm{e}^{-\theta_1}\}^2 \end{pmatrix} \cdot \frac{\mathrm{e}^{\mathrm{i}2\kappa z}}{(\zeta-\overline{\zeta}_1)^2} \tag{A.18}$$

$$\Phi(z,\zeta_1) = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}2\beta_1} \\ -1 \end{pmatrix} \cdot \frac{1}{4}\operatorname{sech}^2\theta_1 \tag{A.19}$$

$$\Psi(z,\zeta_1) = \begin{pmatrix} -\mathrm{e}^{-\mathrm{i}2\beta_1} \\ 1 \end{pmatrix} \cdot \frac{1}{4}\operatorname{sech}^2\theta_1 \tag{A.20}$$

$$\dot\Phi(z,\zeta_1) = \frac{\mathrm{i}}{2k_1}\begin{pmatrix} -\frac{\lambda_1}{\rho}\mathrm{e}^{-\mathrm{i}3\beta_1}\theta_1\operatorname{sech}^2\theta_1 + 2\phi_1^2(z,\zeta_1) - \mathrm{e}^{-\mathrm{i}2\beta_1}\operatorname{sech}\theta_1\mathrm{e}^{\theta_1} \\ \frac{\lambda_1}{\rho}\mathrm{e}^{-\mathrm{i}\beta_1}\theta_1\operatorname{sech}^2\theta_1 + 2\phi_2^2(z,\zeta_1) + \mathrm{e}^{-\mathrm{i}2\beta_1}\operatorname{sech}\theta_1\mathrm{e}^{\theta_1} \end{pmatrix} \tag{A.21}$$

$$\dot\Psi(z,\zeta_1) = \frac{\mathrm{i}}{2k_1}\begin{pmatrix} -\frac{\lambda_1}{\rho}\mathrm{e}^{-\mathrm{i}3\beta_1}\theta_1\operatorname{sech}^2\theta_1 + 2\psi_1^2(z,\zeta_1) + \mathrm{e}^{-\mathrm{i}4\beta_1}\operatorname{sech}\theta_1\mathrm{e}^{-\theta_1} \\ \frac{\lambda_1}{\rho}\mathrm{e}^{-\mathrm{i}\beta_1}\theta_1\operatorname{sech}^2\theta_1 + 2\psi_2^2(z,\zeta_1) - \operatorname{sech}\theta_1\mathrm{e}^{-\theta_1} \end{pmatrix} \tag{A.22}$$

$$\dot\Psi(z,\zeta_1) - \mathrm{i}\frac{1}{k_1}\Psi(z,\zeta_1) = \frac{\mathrm{i}}{2k_1}\begin{pmatrix} -\frac{\lambda_1}{\rho}\mathrm{e}^{-\mathrm{i}3\beta_1}\theta_1\operatorname{sech}^2\theta_1 + \mathrm{e}^{-\mathrm{i}4\beta_1}\operatorname{sech}\theta_1\mathrm{e}^{-\theta_1} \\ \frac{\lambda_1}{\rho}\mathrm{e}^{-\mathrm{i}\beta_1}\theta_1\operatorname{sech}^2\theta_1 - \operatorname{sech}\theta_1\mathrm{e}^{-\theta_1} \end{pmatrix}. \tag{A.23}$$

## References


[1] Hasegawa A 1990 *Optical Solitons in Fibers* 2nd edn (Berlin: Springer)

[2] Aitchinson J S *et al* 1990 Observation of spatial optical solitons in nonlinear glass waveguide *Opt. Lett.* **15** 471–3

[3] Zakharov V E and Shabat A B 1972 Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media *Sov. Phys.–JETP* **37** 364

[4] Zakharov V E and Shabat A B 1973 Interaction between solitons in a stable medium *Sov. Phys.–JETP* **37** 923–30

[5] Kivshar Y S 1993 Dark solitons in nonlinear optics *IEEE J. Quantum Electron.* **29** 250–64 and references therein

[6] Kaup D J 1976 Closure of the squared Zakharov–Shabat eigenstates *J. Math. Anal. Appl.* **54** 849–64
[7] Kaup D J and Newell A C 1978 Solitons as particles, oscillators, and in slowly changing media: a singular perturbation theory *Proc. R. Soc.* A **361** 413–46
[8] Karpman V I 1979 Soliton Evolution in the Presence of Perturbation *Phys. Scr.* **20** 462–78
[9] Kivshar Y S and Malomad B A 1989 Solitons in nearly integrable systems, *Rev. Mod. Phys.* **61** 763–915 and references therein
[10] Keener J P and McLaughlin D W 1977 A Green's function for a linear equation associated with solitons *J. Math. Phys.* **18** 2008–13
[11] Keener J P and McLaughlin D W 1977 Solitons under perturbation *Phys. Rev.* A **16** 777–90
[12] McLaughlin D W and Scott A C 1978 Perturbation analysis of fluxon dynamics *Phys. Rev.* A **18** 1652–80
[13] Kaup D J 1990 Perturbation theory for solitons in optical fibers *Phys. Rev.* A **42** 5689
[14] Herman R L 1990 A direct approach to studying soliton perturbation *J. Phys.* **23A** 2327–62
[15] Yan J R and Tang Y 1996 Direct approach to the study of soliton perturbations *Phys. Rev.* E **54** 6816–24 and references therein
[16] Lamb G L Jr 1980 *Elements of Soliton Theory* (New York: Wiley) 269–71
[17] Konotop V V and Vekslerchik V E 1994 Direct perturbation theory for dark solitons *Phys. Rev.* E **49** 2397–407
[18] Kivshar Y S and Yang X 1994 Perturbation-induced dynamics of dark solitons *Phys. Rev.* E **49** 1657–70
[19] Ikeda H, Matsumoto M and Hasegawa A 1995 Transmission control of dark solitons by means of nonlinear gain *Opt. Lett.* **20** 1113–15
[20] Burtsev S and Camassa R 1997 Nonadiabatic dynamics of dark solitons *J. Opt. Soc. Am.* B **14** 1782–7
[21] Faddeev L D and Takhtajan L A 1987 *Hamiltonian Methods in the Theory of Solitons* (Berlin: Springer)
[22] Huang N N 1996 *Theory of Solitons and Method of Perturbations* (Shanghai: Shanghai Scientific and Technological Eduation Publishing House) (in Chinese)